JOURNAL OF COMPUTATIONAL PHYSICS 75, 498–504 (1988)

Note

Hyperbolic Two-Pressure Models for Two-Phase Flow Revisited

This paper presents corrections to a previous paper (*J. Comput. Phys.* **53** (1984), 124) on stability analyses of one-pressure models for two-phase flow. It also presents some extensions and generalizations of previous work on two-pressure models. The extensions allow both the slopes to the interfaces and the rates of mass transfer through the interfaces to be non-negligible. These enlargements of the domain of applicability of the two-pressure models introduce extra derivative terms into the models. These extra derivative terms do not change the basic character of the model. The two-pressure models remain stable in the sense of von Neumann a.e. in state space with the more complete modelling of the interface. © 1988 Academic Press, Inc.

1. Introduction, Background, and Errata

Reference [1] introduced the two-pressure models to this journal and showed that they were stable in the sense of von Neumann a.c. in state space; also it reviewed some results of Wendroff [2] on the instability of the one-pressure models. Wendroff [3] recently pointed out an error in that review.

In the equaton for the mixture sound speed, c_m (on the bottom of p. 139 in Ref. [1]), the denominator in that expression was incorrectly printed as $\rho_1 c_1^2 + \rho_2 c_2^2$. Here ρ_n for $n = 1, 2$ is the averaged value of the mass density of phase n and c_n is the sound speed of phase n for $n = 1, 2$.

The correct expression for the denominator is $\alpha_2 \rho_1 c_1^2 + \alpha_1 \rho_2 c_2^2$, where α_n is the volume fraction of phase n.

In summary, the correct equation for the mixture sound speed in the one-pressure models is

$$\rho_m c_m^2 = \rho_1 c_1^2 \rho_2 c_2^2 / (\alpha_2 \rho_1 c_1^2 + \alpha_1 \rho_2 c_2^2),$$

where ρ_m is the mixture density given by $\rho_m = \alpha_1 \rho_1 + \alpha_2 \rho_2$.

Shortly after reference [1] appeared the review article on two-phase flow by Stewart and Wendroff [4] appeared. There seemed to be a discrepancy between the conditions for real roots of the characteristic polynomial for the single-pressure model. However, with this correction in the equation for the mixture sound speed, the W-inequality on page 142 of Ref. [1] is equivalent to the inequalities on page 407 of Stewart and Wendroff [4]; these inequalities give the constraints for the

0021-9991/88 $3.00

characteristic roots to be real. Thus, there is no disagreement between the two papers.

There are two other errors in Ref. [1]:

- In Eq. (2.78) on page 137 the third "ρ_n" in that equation should be replaced with a "p_n";
- On page 140 in the equation for a_0 the third "w" should be replaced with a "2."

In reference [1] the two-pressure models, 5E2P and 8E2P, were analyzed for stability in the sense of von Neumann under certain simplifying assumptions:

(i) separated, stratified flow between plates at $y=0$ and $y=H$ with Phase 1 in the region $0<y<Y$ and Phase 2 in the region $Y<y<H$;

(ii) interface slope small, i.e., $|\partial Y/\partial x| \approx 0$;

(iii) negligible rates of mass transfer across the interface, i.e., $\dot{m} \approx 0$.

This brief paper gives a quick sketch of the relaxation of some of these constraints:

On (iii) the extension allows any finite value of $\dot{m}$.

On (ii) the extension allows any finite value of $\partial Y/\partial x$.

On (i) we have done some preliminary work on the extensions to multilayered flow and bubbly flow which the interested reader may find in Ref. [5]; however, we shall not discuss these latter extensions in this brief report.

Because the combination of extensions (ii) and (iii) introduces new derivative terms into the models it is not immediately obvious that the two-pressures models would remain stable in the sense of von Neumann a.e. in state space. However, as we shall show, they do.

2. Notation and Nomenclature

We follow the same notation and nomenclature as Ref. [1] and therefore only briefly review it. The planar, two-dimensional conservation (or balance) laws for a single phase in the spatial (Eulerian) reference frame are

$$D\mathbf{f} + \partial\mathbf{F}/\partial x + \partial\mathbf{G}/\partial y = \mathbf{S},$$

where

$$D\mathbf{f} = \partial\mathbf{f}/\partial\mathbf{t} + \partial u\mathbf{f}/\partial x + \partial v\mathbf{f}/\partial y$$

$$\mathbf{f} = (\rho, \rho u, \rho v, \rho E)^{\mathrm{T}}$$

$$\mathbf{F} = (0, p, 0, up)^{\mathrm{T}}$$

$$\mathbf{G} = (0, 0, p, vp)^{\mathrm{T}}$$
$$\mathbf{S} = (S^{\rho}, S^{\rho u}, S^{\rho v}, S^{\rho E})^{\mathrm{T}}.$$

Here (x, y) are spatial coordinates, t is the temporal coordinate, (u, v) is the fluid velocity vector, ρ is the mass density, E is the specific total energy, and $\mathbf{S}$ is the vector of source functions.

We consider the case of planar, stratified flow between plates at $y = 0$ and $y = H$. Phase 1 is in the region $0 < y < Y$ and Phase 2 is in the region $Y < y < H$. $Y = Y(x, t)$ is the interface between Phase 1 and Phase 2.

The volume fraction of phase n is denoted by α_n. Under the assumption that there are no detachments, the conservtion of volume is expressed by

$$\alpha_1 + \alpha_2 = 1$$

and we also have the relation

$$Y = \alpha_1 H.$$

Boundary values. The value of f at $y = 0$ is denoted by f_0 and at $y = H$ is denoted by f_H. The value of f on the n-side of the interface is denoted by $\hat{f}_n$.

Point functions. Before averaging we have the point functions, $f_{\cdot n}$, defined by

$$f_{\cdot n}(x, y, t) = \begin{cases} f(x, y, t) \text{ if } (x, y, t) & \text{is in phase } n \\ 0 & \text{if not.} \end{cases}$$

Averaging operators. The averaging operators are defined by

$$A_n(\cdot) = \int_0^H (\cdot)\, dy/(\alpha_n H).$$

Average-value functions. The average-value functions are denoted by f_n and defined by

$$f_n = A_n(f_{\cdot n}).$$

Interfacial mass transfer. The mass/area/second flowing through the interface will be denoted by $\dot{m}$. In Ref. [1] the equations are only correct for $\dot{m} \approx 0$.

Velocity of the interface. It appears that some discussions about interface conditions and interface mass transfer have been obscured because the phrase "velocity of the interface" is somewhat ambiguous; "tangential velocity of the interface" in particular, seems to be open to many different definitions. Here we take the velocity in the direction normal to the interface in the spatial reference frame; we refer to this as the normal Eulerian velocity of the interface.

It can be shown [5] that the normal Eulerian velocity of the interface is given by

$$\mathbf{n} = (-\partial Y/\partial x, 1)(\partial Y/\partial t)/M^2,$$

where

$$M=(1+(\partial Y/\partial x)^2)^{1/2}.$$

Note that

$$\|\mathbf{n}\| = |\partial Y/\partial t|/M,$$

where $\|\cdot\|$ is the usual, Euclidean norm.

If an interface with normal Eulerian velocity $\mathbf{n}$ is traveling through a field governed by the equation

$$\partial u/\partial t + \partial f/\partial x + \partial g/\partial y = S$$

(where S has no derivative terms) then it can be shown (e.g., see [6]) that the jump in the field across the interface satisfies the general jump relation

$$\|\mathbf{n}\|^2 [u] = [(f, g)] \cdot \mathbf{n},$$

where $[f]$ is the jump in f across the interface, i.e., $[f] = \hat{f}_2 - \hat{f}_1$.

Mass Transfer Relations across the Interface

Applying the general jump relation to the conservation of mass equation yields

$$\|\mathbf{n}\|^2 [\rho] = [(\rho u, \rho v)] \cdot \mathbf{n}.$$

Let v_n be the fluid velocity on the n-side of the interface measured in the direction of the normal Eulerian velocity of the interface, i.e.,

$$v_n = (\hat{u}_n, \hat{v}_n) \cdot \mathbf{n}/\|\mathbf{n}\|.$$

In terms of v_n we (see [5]) can derive the following expression for the conservation of the mass passing through the interface

$$\dot{m} = \hat{\rho}_1(\|\mathbf{n}\| - v_1) = \hat{\rho}_2(\|\mathbf{n}\| - v_2).$$

Observe that from the last two equations the equation for the transverse velocity of phase n at the interface follows:

$$\hat{v}_n = H(\partial\alpha_1/\partial t + \hat{u}_n\, \partial\alpha_1/\partial x) - \dot{m}M/\hat{\rho}_n.$$

Note that when $\dot{m}=0$ this reduces to Stewart and Wendroff's [4] interface conditions (2.1.7) and (2.1.8).

Other Jump Relations at the Interface That Follow from the General Jump Relation

$$\dot{m}[u] = [p]\, n_1/\|\mathbf{n}\|$$
$$\dot{m}[v] = [p]\, n_2/\|\mathbf{n}\|$$
$$\dot{m}[v] = [p]$$
$$\dot{m}[V] = -[v]$$
$$\dot{m}[E] = [vp]$$
$$0 = [e] + \bar{p}[V],$$

where e is the specific internal energy, V is the specific volume and

$$\bar{p} = (\hat{p}_1 + \hat{p}_2)/2.$$

3. Generalizations of the Interface Conditions

In Ref. [1] we showed that the averaged equations have the form

$$D_n(\alpha_n \mathbf{f}_n) + \partial(\alpha_n \mathbf{F}_n)/\partial x - \hat{\mathbf{F}}_n \partial\alpha_n/\partial x + \Delta_n(\mathbf{G}_n)/H = \$_n,$$

where $\$_n$ is the generalized source term given by

$$\$_n = \alpha_n \mathbf{S}_n - \mathbf{B}_n(\mathbf{f}).$$

When nonnegligible mass transfer and nonnegligible interface slope are included the $\mathbf{B}_n$ becomes

$$\mathbf{B}_1(\mathbf{f}) = -[\mathbf{f}_0 v_0 + \dot{m}M\hat{\mathbf{f}}_1/\hat{\rho}_1]/H$$

and

$$\mathbf{B}_2(\mathbf{f}) = +[\mathbf{f}_H v_H + \dot{m}M\hat{\mathbf{f}}_2/\hat{\rho}_2]/H.$$

Note that in the expressions for $\mathbf{B}_n$ in Ref. [1] no M appears because when $\partial Y/\partial x$ is small then $M \approx 1$.

Observe that since

$$M^2 = 1 + H^2(\partial\alpha_1/\partial x)^2,$$

the inclusion of the $\dot{m}M$ term introduces a $\partial\alpha_1/\partial x$ term which could possibly change the character of the equations. Therefore, we investigate the effect this has on the von Neumann stability of the two-pressure models.

Note that if $\partial\alpha_n/\partial x$ has large magnitude then

$$\mathbf{B}_n(\mathbf{f}) \approx (-1)^n\, (\dot{m}\hat{\mathbf{f}}_n/\hat{\rho}_n)\, \sigma_n \partial\alpha_n/\partial x$$

where

$$\sigma_n = \mathrm{sign}(\partial \alpha_n / \partial x)$$

for $n = 1, 2$.

Recall from Ref. [1] that the analysis of the 5E2P model reduced to studying the von Neumann stability of a system

$$\partial \mathbf{U}/\partial t + \mathbf{A}\, \partial \mathbf{U}/\partial x = 0,$$

where

$$\mathbf{U}^T = (\alpha_2 \rho_2, \alpha_2 \rho_2 u_2, \alpha_1 \rho_1, \alpha_1 \rho_1 u_1, \alpha_1)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & A_{15} \\ c_2^2 - u_2^2 & 2u_2 & 0 & 0 & A_{25} \\ 0 & 0 & 0 & 1 & A_{35} \\ 0 & 0 & c_1^2 - u_1^2 & 2u_1 & A_{45} \\ 0 & 0 & 0 & 0 & \hat{u} \end{bmatrix}$$

with

$$A_{15} = 0 = A_{35}$$

$$A_{25} = -P_2$$

$$A_{45} = +P_1$$

and

$$P_n = p_n - \hat{p} - c_n^2 \rho_n$$

when the $\dot{m}M$ term is neglected.

When the $\dot{m}M$ term is not neglected and $\partial \alpha_n / \partial x$ has large magnitude then

$$A_{15} \approx -\dot{m}\sigma_2$$

$$A_{25} \approx -(\dot{m}\sigma_2 u_2 + P_2)$$

$$A_{35} \approx +\dot{m}\sigma_1$$

$$A_{45} \approx +(\dot{m}\sigma_1 u_1 + P_1).$$

Note that the values of A_{15}, $1 \leqslant i \leqslant 4$, have no effect on the eigenvalues of $\mathbf{A}$. Therefore, as in Ref. [1], we have five real eigenvalues $u_n \pm c_n$ and $\hat{u}$. The analysis for the eigenvectors follows the same pattern with very similar results as in [1]. The 5E2P analysis extends readily to the 8E2P model just as in [1].

Thus, we arrive at the conclusion: the two-pressure models for two-phase flow continue to be stable in the sense of von Neumann a.e. in state space when the effects of nonnegligible mass transfer through the interface and nonnegligible slope to the interface are included.

Acknowledgments

We again wish to acknowledge the advice and encouragement of B. Wendroff of Los Alamos National Laboratory. Also we wish to thank J. Trapp of the University of Colorado at Denver for many stimulating discussions. Finally, we wish to thank M. Kriebel and V. Mousseau (students at Michigan Technological University) for carefully reading Ref. [1] and this paper.

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Received: March 20, 1987; revised May 29, 1987

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